§ Comparison theorems in Riemannian Geometry

Idea: curvature bdd $\Rightarrow$ geometric bdd

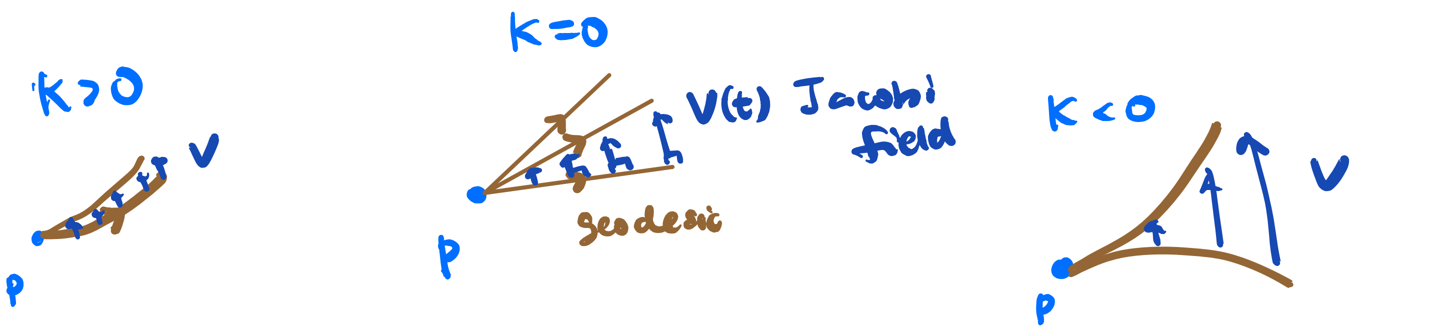
in comparison with spaces of constant curvature $(\mathbb{H}^n, \mathbb{R}^n, \mathbb{S}^n)$.

One example is Bonnet-Myers Thm:

$$\underset{\text{complete}}{(M^n, g)} \quad \mathrm{Ric}^{M^n} \geq \mathrm{Ric}^{\mathbb{S}^n} \quad \Rightarrow \quad \mathrm{diam}(M, g) \leq \mathrm{diam}(\mathbb{S}^n)$$

We will discuss another comparison theorem due to Rauch.

Idea: Curvature $\rightsquigarrow$ spreading of geodesic $\leftrightsquigarrow$ Jacobi field estimates



$|V|$ small
$\Leftrightarrow$ geodesics converges

$|V|$ big
$\Leftrightarrow$ geodesic diverges

The following proposition gives a quantitative description of the effect of spreading of geodesics by curvature via the length of Jacobi fields.

Prop: Let $\gamma : [0,a] \to (M^n, g)$ be a geodesic w/. $\gamma(0) = p$, $\gamma'(0) = v$.

For any $w \in T_v(T_pM)$, $|w| = 1$, consider the Jacobi field

$$V(t) := (d\exp_p)_{tv}(tw)$$

Note: $V(0) = 0$, $V'(0) = w$

$V'' + R(\gamma', V)\gamma' = 0$

THEN, we have the following Taylor expansion (near $t=0$)

$$|V(t)|^2 = t^2 - \frac{1}{3}\overset{M}{R}(v,w,v,w)\, t^4 + o(t^4) \quad \text{as } t \to 0$$

Proof: We do Taylor expansion for the function $f(t) := |V(t)|^2$.

$$f(0) = |V(0)|^2 = 0 \qquad (\because V(0) = 0)$$

$$f'(t) = 2\langle V(t), V'(t)\rangle \overset{t=0}{\Rightarrow} f'(0) = 0 \qquad (\because V(0) = 0)$$

$$f''(t) = 2\langle V', V'\rangle + 2\langle V, V''\rangle \overset{\text{at } t=0}{\Longrightarrow} f''(0) = 2|V'(0)|^2 = 2$$

$$= 2|V'|^2 - 2R(\gamma', V, \gamma', V)$$

$$f'''(t) = 4\langle V', \underbrace{V''}_{= -R(\gamma', V)\gamma'}\rangle - 2(\nabla_{\gamma'}R)(\gamma', V, \gamma', V) \overset{\text{at } t=0}{\Longrightarrow} f'''(0) = 0$$

$$- 4R(\gamma', V, \gamma', V')$$

$$f''''(0) = -4R(\gamma', V', \gamma', V') - 4R(\gamma', V', \gamma', V') = -8R(v,w,v,w)$$

▫

Cor: When $w \perp v$, then

$$|V(t)| = t - \frac{1}{6}K(\operatorname{span}\{v,w\})\, t^3 + o(t^3)$$

Note: $K \le \tilde{K}$

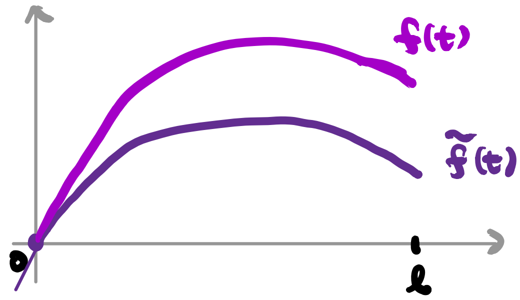
$\Rightarrow |V(t)| \ge |\tilde{V}(t)|$

for small $t \approx 0$

Q: What about for large $t \gg 0$? **A:** Rauch comparison thm.

2D | "Sturm's" Theory for ODE

2D: $V(t) = f(t)\, J\gamma'(t)$; $\tilde{V}(t) = \tilde{f}(t)\, J\tilde{\gamma}'(t)$



$$\begin{cases} f''(t) + K(t) f(t) = 0, & t \in [0,\ell] \\ f(0) = 0 \end{cases}$$

$$\begin{cases} \tilde{f}''(t) + \tilde{K}(t) \tilde{f}(t) = 0, & t \in [0,\ell]. \\ \tilde{f}(0) = 0 \end{cases}$$

Suppose $f'(0) = \tilde{f}'(0) > 0$, $\tilde{f}(t) \neq 0 \quad \forall t \in [0,\ell]$.

and $\tilde{K}(t) \geq K(t)$

THEN : $\tilde{f}(t) \leq f(t) \quad \forall t \in [0,\ell]$

Note: We need to be more careful for higher dimensions.

We first establish a useful lemma which says that (normal) Jacobi fields minimize the index form among all (normal) vector fields with the same boundary values.

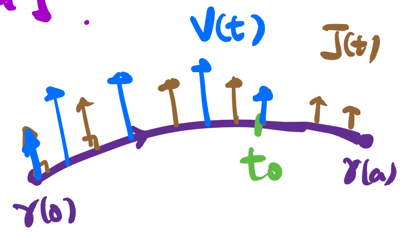
Index Lemma: Let $\gamma : [0,a] \to M^n$ be a geodesic.

Let $J(t)$ be a normal Jacobi field along γ

Suppose $V(t)$ be any (piecewise smooth) normal vector field along γ

s.t. $J(0) = V(0)$ & $J(t_0) = V(t_0)$ for some $t_0 \in (0,a]$

ASSUME: γ has NO conjugate points on $[0,a]$.



THEN : $\boxed{I_{t_0}(J,J) \leq I_{t_0}(V,V)}$

and "=" holds $\iff J(t) \equiv V(t)$ on $[0, t_0]$.

Here. $I_{t_0}(W,W) := \int_0^{t_0} |W'|^2 - R(\gamma', W, \gamma', W)\, dt$

Proof of Index Lemma:

Let $\mathcal{J} := \{ J(t)$ normal Jacobi fields along γ on $[0,a]$ s.t. $J(0)=0 \}$

which is a vector space of dim $= n-1$. Fix a basis $\{J_1, \dots, J_{n-1}\}$.

$\nexists$ conjugate pt along γ on $[0,a]$ $\Rightarrow \forall t \in [0,a]$, $\{J_1(t), \dots, J_{n-1}(t)\} \underset{\text{basis}}{\subseteq} T_{\gamma(t)}M$

We can write $V(t) = \sum_{i=1}^{n-1} \underbrace{f_i(t)}_{\text{piecewise smooth}} J_i(t)$; $J(t) = \sum_{i=1}^{n-1} \underbrace{\alpha_i}_{\text{constants}} J_i(t)$

Claim: $\langle V', V' \rangle - R(\gamma', V, \gamma', V)$ integrand of $I(V,V)$

$$= \langle \sum_i f_i' J_i , \sum_j f_j' J_j \rangle + \frac{d}{dt} \left(\langle \sum_i f_i J_i , \sum_j f_j J_j' \rangle \right)$$

Pf of Claim: Recall: $J_i'' = -R(\gamma', J_i)\gamma'$

$$\text{L.H.S.} = \underbrace{\langle \sum_i (f_i' J_i + f_i J_i'), \sum_j (f_j' J_j + f_j J_j') \rangle}_{\text{expand this}} - \sum_{i,j} f_i f_j \underbrace{R(\gamma', J_i, \gamma', J_j)}_{= -\langle J_i'', J_j \rangle}$$

$$= \sum_{i,j} \Big(f_i' f_j' \langle J_i, J_j \rangle + 2 f_i' f_j \langle J_i, J_j' \rangle + f_i f_j \langle J_i', J_j' \rangle + f_i f_j \langle J_i'', J_j \rangle \Big)$$

2nd term of R.H.S. $= \sum_{i,j} \Big(\langle f_i' J_i + f_i J_i', f_j J_j' \rangle + \langle f_i J_i, f_j' J_j' + f_j J_j'' \rangle \Big)$

$$= \sum_{i,j} \Big(f_i' f_j \langle J_i, J_j' \rangle + f_i f_j \langle J_i', J_j' \rangle + f_i f_j' \langle J_i, J_j' \rangle + f_i f_j \langle J_i, J_j'' \rangle \Big)$$

Need: $\langle J_i', J_j \rangle = \langle J_i, J_j' \rangle \quad \forall t \in [0,a]$

Reason: Let $h(t) := \langle J_i', J_j \rangle - \langle J_i, J_j' \rangle$.

Note $h(0) = 0$ since $J_i(0) = 0 \quad \forall i$.

AND: $h'(t) = \langle J_i'', J_j \rangle + \cancel{\langle J_i', J_j' \rangle} - \cancel{\langle J_i', J_j' \rangle} - \langle J_i, J_j'' \rangle$

$$= R(\gamma', J_i, \gamma', J_j) - R(\gamma', J_j, \gamma', J_i) \equiv 0$$

Apply the claim to $V(t)$ and $J(t)$.

$$I_{t_0}(V,V) = \int_0^{t_0} \overbrace{\langle \sum_i f_i' J_i, \sum_j f_j' J_j \rangle}^{= \|\sum_i f_i' J_i\|^2 \geq 0} dt + \underline{\langle \sum_i f_i J_i, \sum_j f_j J_j' \rangle (t_0)}$$

$$I_{t_0}(J,J) = \underline{\langle \sum_i \alpha_i J_i, \sum_j \alpha_j J_j' \rangle (t_0)}$$

$\because V(t_0) = J(t_0)$

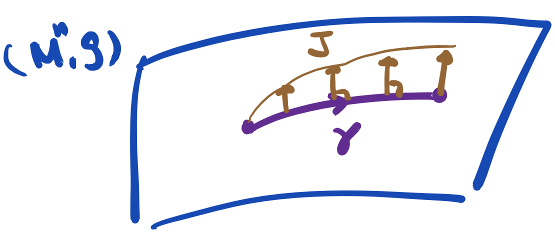
$\Leftrightarrow f_i(t_0) = \alpha_i$

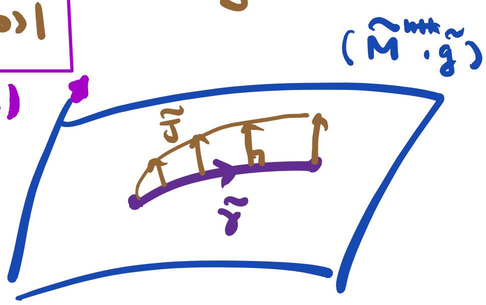
$\square$

Rauch comparison theorem

Let

Riem. mfd	(M^n, g)	ASSUME	$(\tilde{M}^{n+k}, \tilde{g})$
geodesics	$\gamma : [0,a] \to M$	$\boxed{\begin{array}{c} \lvert\gamma'\rvert = \lvert\tilde{\gamma}'\rvert \\ J(0) = \tilde{J}(0) = 0 \\ \lvert J'(0)\rvert = \lvert\tilde{J}'(0)\rvert \end{array}}$	$\tilde{\gamma} : [0,a] \to \tilde{M}$
Jacobi fields (normal)	J	(normalization)	$\tilde{J}$





ASSUME: (i) $\nexists$ conjugate pts along $\tilde{\gamma}$ on $[0,a]$

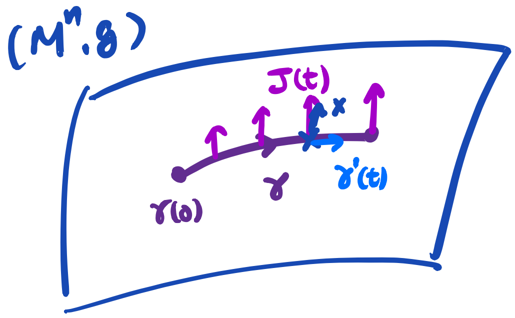
(ii) $K_{\gamma(t)}(\mathrm{span}\{\gamma'(t), x\}) \leq \tilde{K}_{\tilde{\gamma}(t)}(\mathrm{span}\{\tilde{\gamma}'(t), \tilde{x}\})$

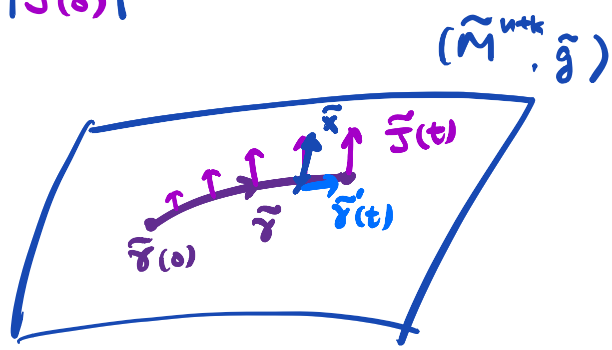
$\forall t \in [0,a], \ \forall x \in T_{\gamma(t)}M, \ \tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$

THEN: $|\tilde{J}(t)| \le |J(t)|$ for **ALL** $t \in [0,a]$.

Rauch comparison theorem (restated)

Let
- (M^n, g), $(\tilde{M}^{n+k}, \tilde{g})$ be complete Riem. manifolds
- $\gamma : [0,a] \to M$, $\tilde{\gamma} : [0,a] \to \tilde{M}$ be geodesic in M and $\tilde{M}$
  s.t. $|\gamma'| = |\tilde{\gamma}'|$ (ie. same speed)
- $J, \tilde{J}$ are normal Jacobi fields along $\gamma, \tilde{\gamma}$ respectively
  s.t. $J(0) = 0 = \tilde{J}(0)$, $|J'(0)| = |\tilde{J}'(0)|$





ASSUME: ① $\nexists$ conjugate pt along $\tilde{\gamma}$ on $[0,a]$

* ② $\forall t \in [0,a]$, $\forall x \in T_{\gamma(t)}M$, $\forall \tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$,

we have $K_{\gamma(t)}(\mathrm{span}\{\gamma'(t), x\}) \le \tilde{K}_{\tilde{\gamma}(t)}(\mathrm{span}\{\tilde{\gamma}'(t), \tilde{x}\})$

THEN: $|J(t)| \ge |\tilde{J}(t)|$ $\forall t \in [0,a]$

Recall: we proved the

Index Lemma: On a geodesic segment without conjugate points,

$$I(J,J) \le I(V,V)$$ where J and V agree at the end points and J is a Jacobi field.

Proof of Rauch comparison Thm:

WLOG, assume $|J'(0)| = |\tilde{J}'(0)| > 0$.

Denote: $v(t) := |J(t)|^2$ and $\tilde{v}(t) := |\tilde{J}(t)|^2$

Goal: $v(t) \ge \tilde{v}(t) \quad \forall t \in [0,a]$

Observe, by assumption ①, $\tilde{v}(t) > 0 \quad \forall t \in (0,a]$. So we define

$$h(t) := \frac{v(t)}{\tilde{v}(t)} \qquad t \in (0,a]$$

Goal: $h(t) \ge 1 \quad \forall t \in (0,a]$

$v'(t) = 2\langle J(t), J'(t)\rangle$

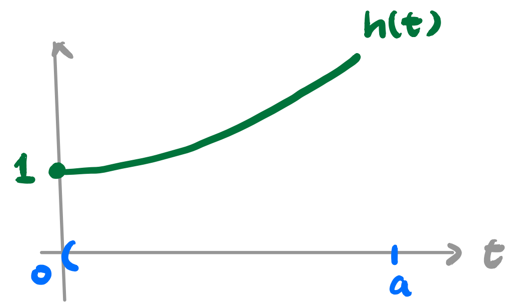
$v''(t) = 2\langle J'(t), J'(t)\rangle + 2\langle J(t), J''(t)\rangle$

By L'Hospital's Rule,

$$\lim_{t\searrow 0} h(t) = \lim_{t\searrow 0} \frac{v'(t)}{\tilde{v}'(t)} = \lim_{t\searrow 0} \frac{v''(t)}{\tilde{v}''(t)} = \frac{|J'(0)|^2}{|\tilde{J}'(0)|^2} = 1$$

Claim: $h'(t) \ge 0 \quad \forall t \in (0,a)$

i.e. $v'\tilde{v} \ge v\tilde{v}' \quad \forall t \in (0,a)$ (*)



Pf of Claim: Fix any $t_0 \in (0,a)$.

Case 1: $v(t_0) = 0 \quad (\Rightarrow J(t_0) = 0)$

Then, $v'(t_0) = 2\langle J(t_0), J'(t_0)\rangle = 0$

Clearly, (*) holds.

Case 2: $v(t_0) \neq 0 \quad (\Rightarrow \quad v(t_0) = |J(t_0)|^2 > 0)$

By rescaling, we obtain two new Jacobi fields

$$U(t) := \frac{1}{\sqrt{v(t_0)}} J(t) \quad ; \quad \tilde{U}(t) := \frac{1}{\sqrt{\tilde{v}(t_0)}} \tilde{J}(t)$$

Observe: $\dfrac{v'(t_0)}{v(t_0)} = 2\, I_{t_0}(U,U)$ AND $\dfrac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = 2\, I_{t_0}(\tilde{U},\tilde{U})$

Why? $2\, I_{t_0}(U,U) := 2\int_0^{t_0} \langle U',U'\rangle - R(\gamma',U,\gamma',U)\, dt$

$$= \int_0^{t_0} (\langle U,U\rangle)''\, dt \overset{\text{by}}{\underset{\text{part}}{=}} \langle U,U\rangle'(t_0)$$

$$= 2\langle U(t_0), U'(t_0)\rangle = 2\,\frac{\langle J(t_0), J'(t_0)\rangle}{v(t_0)} = \frac{v'(t_0)}{v(t_0)}$$

So, it suffices to show

$$\boxed{I_{t_0}(\tilde{U},\tilde{U}) \leq I_{t_0}(U,U)} \quad (*)$$

Fix parallel O.N.B. along γ, $\tilde{\gamma}$:

$$\left\{ e_1 = \frac{\gamma'}{|\gamma'|}, e_2, \dots, e_n \right\}(t) \text{ for } T_{\gamma(t)}M^n \text{ s.t. } e_2(t_0) = U(t_0)$$

$$\left\{ \tilde{e}_1 = \frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}, \tilde{e}_2, \dots, \tilde{e}_{n+k} \right\}(t) \text{ for } T_{\tilde{\gamma}(t)}\tilde{M}^{n+k} \text{ st } \tilde{e}_2(t_0) = \tilde{U}(t_0)$$

Define a map $\phi : \begin{Bmatrix} \text{vector field} \\ \text{along } \gamma(t) \end{Bmatrix} \to \begin{Bmatrix} \text{vector field} \\ \text{along } \tilde{\gamma}(t) \end{Bmatrix}$

$$V(t) = \sum_{i=1}^{n} g_i(t)\, e_i(t) \overset{\phi}{\longmapsto} (\phi V)(t) := \sum_{i=1}^{n} g_i(t)\, \tilde{e}_i(t)$$

same coefficients

Properties of ϕ :

- $\langle \phi V, \phi V \rangle = \langle V, V \rangle \quad \forall t \in [0,a]$
- $(\phi V)' = \phi(V') \quad \forall t \in [0,a]$
- * $I_{t_0}(\phi V, \phi V) \leq I_{t_0}(V, V)$

Reason:

$$I_{t_0}(V,V) := \int_0^{t_0} |V'(t)|^2 - R(\gamma', V, \gamma', V)(t)\, dt$$

$$\geq \qquad \| \text{ by first 2 properties} \qquad \geq \text{ by } 1^{st} \text{ property and } K \leq \tilde{K}$$

$$I_{t_0}(\phi V, \phi V) := \int_0^{t_0} |(\phi V)'|^2 - \tilde{R}(\tilde{\gamma}', \phi V, \tilde{\gamma}', \phi V)\, dt$$

Recall by construction, we have

$$(\phi \mho)(t_0) = \phi(e_2(t_0)) = \tilde{e}_2(t_0) = \tilde{\mho}(t_0)$$

i.e. $\phi \mho$ and $\tilde{\mho}$ have the same boundary values along $\tilde{\gamma}$ on $[0, t_0]$

By (#) and Index Lemma,

$$\frac{1}{2}\frac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = I_{t_0}(\tilde{\mho}, \tilde{\mho}) \leq I_{t_0}(\phi\mho, \phi\mho) \overset{(\#)}{\leq} I_{t_0}(\mho, \mho) = \frac{1}{2}\frac{v'(t_0)}{v(t_0)}$$

$\square$

Volume comparison via Jacobi field estimates

Thm: (Bishop-Gromov & Gunther)

Let (M^n, g) be a complete Riem. mfd.

Fix $p \in M$, let $a, b \in \mathbb{R}$ be constants.

(i) If $Ric^M_x(u,u) \geq (n-1)\, a \quad \forall x \in M,\ \forall u \in T_xM,\ |u| = 1,$

then $$Vol(B^M_p(r)) \leq Vol(B^{\mathcal{M}_a}(r)) \quad \forall r \in (0, inj_p(M)).$$

Here, $B^{\mathcal{M}_a}(r)$ = Volume of ball of radius r inside a space form $\mathcal{M}_a$ of const. curvature a

(ii) If $K^M \leq b$, then

$$Vol(B^M_p(r)) \geq Vol(B^{\mathcal{M}_b}(r)) \quad \forall r \in (0, inj_p(M)).$$

Digression: Jacobi field and Volume estimates

Volume form of (M^n, g) : $\quad dV_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$

in local coord. $(x^1, \ldots, x^n)$.

and $\quad Vol(\mathcal{U}) = \int_{\mathcal{U}} \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n$ $\quad (\mathcal{U} \subseteq M)$

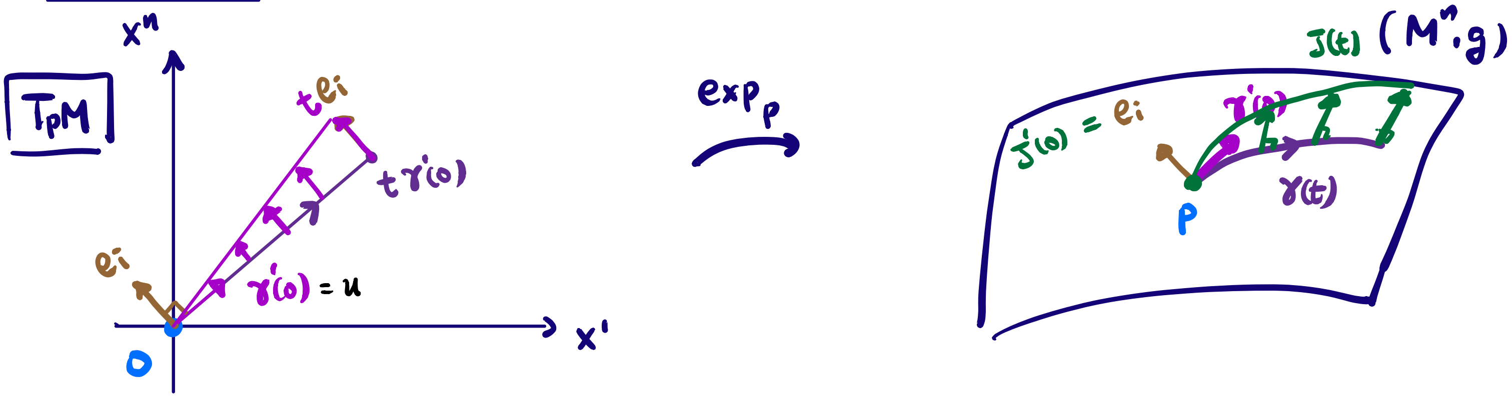
In particular, in geodesic normal coordinates (based at $p \in M$)

$$Vol(B^M_p(r)) = \int_{S^{n-1}} \int_0^r J(u,t) \cdot t^{n-1}\, dt\, du$$

where $\quad J(u,t) := t^{-(n-1)} \sqrt{\det \langle \underbrace{J_i(t)}, \underbrace{J_j(t)} \rangle}$

some Jacobi fields

Picture:



Fix $u \in T_pM$, $|u| = 1$, and a geodesic $\gamma(t) = \exp_p(tu)$

Fix an O.N.B. $\{u = e_1, e_2, \dots, e_n\}$ of T_pM

Let $J_i(t)$ be the Jacobi field along $\gamma(t)$ s.t

$$J_i(0) = 0 \quad \text{and} \quad J_i'(0) = e_i$$

i.e $$J_i(t) = d(\exp_p)_{tu}(te_i)$$

So, $$dV_g = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n$$

$$= \sqrt{\det\big(\underbrace{d\exp(e_i)}_{\frac{1}{t}J_i(t)}, \underbrace{d\exp(e_j)}_{\frac{1}{t}J_j(t)}\big)}\; t^{n-1}\, dt\, du$$

$$= J(u,t)\, t^{n-1}\, dt\, du$$

On the other hand, we have explicit formula for the Jacobi fields in space forms $\mathcal{M}_a$

parallel v.f.

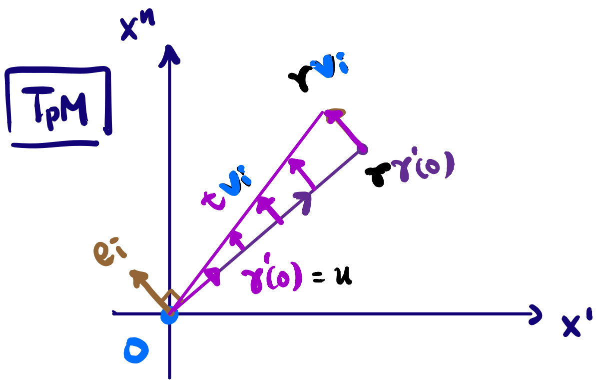
<u>Jacobi eq.^n</u>: $$J'' + aJ = 0 \Rightarrow J(t) = S_a(t)\, E(t)$$

and $J(0) = 0$

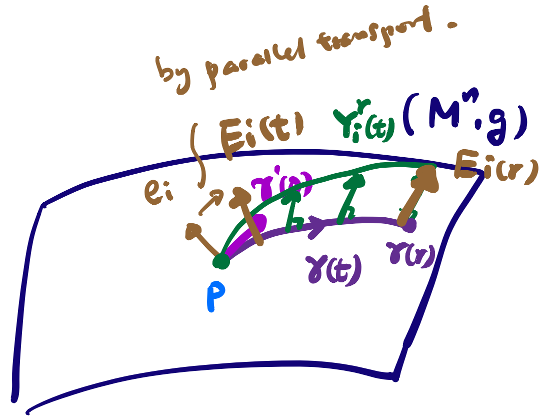
where $$S_a(t) := \begin{cases} \sin\sqrt{a}\, t & , \text{ if } a > 0 \\ t & , \text{ if } a = 0 \\ \sinh\sqrt{-a}\, t & , \text{ if } a < 0 \end{cases}$$

Proof of Volume Comparison Thm (i) : $Ric \geq (n-1)a$

Picture:



$\xrightarrow{\exp_p}$



Denote $Y_i^r(t)$ to be the unique Jacobi field s.t.

$$Y_i^r(0) = 0 \quad \text{and} \quad Y_i^r(r) = E_i(r)$$

and $V_i \in T_pM$ s.t. $d(\exp_p)_{tu}(tV_i) = Y_i^r(t)$

Note: $J(u,t) = C_r\, t^{1-n} \det\left(Y_2^r(t), \dots, Y_n^r(t)\right)$

(Ex) where $C_r = \left[\det\left(Y_2^{r\prime}(0), \dots, Y_n^{r\prime}(0)\right)\right]$

Denote: For each $u \in T_pM$, $|u| = 1$,

$$f(t) := J(u,t)$$

Lemma: $$\frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I_r(Y_i^r, Y_i^r) - \frac{n-1}{r}$$

Proof of Lemma:

Let $g(t) := \det\left(\langle Y_i^r(t), Y_j^r(t)\rangle\right)$. Then

$$f(t) := J(u,t) := C_r\, t^{1-n} \det\left(Y_2^r(t), \dots, Y_n^r(t)\right) = C_r\, t^{1-n}\sqrt{g(t)}$$

$$\Rightarrow \quad f'(t) = C_r\, t^{1-n} \cdot \frac{1}{2} \frac{g'(t)}{\sqrt{g(t)}} - C_r (n-1)\, t^{-n} \sqrt{g(t)}$$

$$\Rightarrow \quad \frac{f'(t)}{f(t)} = \frac{g'(t)}{2g(t)} - \frac{n-1}{t}$$

On the other hand, we want $\displaystyle \frac{g'(r)}{2g(r)} = \sum_{i=2}^{n} I_r(Y_i^r, Y_i^r)$.

<u>Note</u>: $g(r) := \det\left(\langle Y_i^r(r), Y_j^r(r)\rangle\right) = \det \langle E_i(r), E_j(r)\rangle = 1$

and $\displaystyle g'(r) = 2 \sum_{i=2}^{n} \langle (Y_i^r)'(r), Y_i^r(r)\rangle$

By the proof of Claim of Index Lemma, $I_r(Y_i^r, Y_i^r) = \langle Y_i^r(r), Y_i^{r\prime}(r)\rangle$.

□ Lemma

To finish the proof of Volume Comparison Thm (i),

Consider the vector field along γ

$$X_i^r(t) := \frac{S_a(t)}{S_a(r)} E_i(t)$$

s.t. $X_i^r(0) = 0 = Y_i^r(0)$ and $X_i^r(r) = E_i(r) = Y_i^r(r)$.

By Index Lemma, $I_r(Y_i^r, Y_i^r) \le I_r(X_i^r, X_i^r)$.

<u>Note</u>: $\displaystyle I_r(X_i^r, X_i^r) = \int_0^r \left(\frac{S_a'(t)}{S_a(r)}\right)^2 - \left(\frac{S_a(t)}{S_a(r)}\right)^2 R(\gamma', E_i, \gamma', E_i)\, dt$

$$= \int_0^r \left(\frac{S_a(t)}{S_a(r)}\right)^2 \underbrace{[a - R(\gamma', E_i, \gamma', E_i)]}_{\sum_{i=1}^{n}(\ \ ) \le 0 \text{ by assumption}} dt + \frac{S_a'(r)}{S_a(r)}$$

$$\Rightarrow \quad \frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I_r(Y_i^r, Y_i^r) - \frac{n-1}{r} \le \sum_{i=2}^{n} I_r(X_i^r, X_i^r) - \frac{n-1}{r}$$

$$\| \qquad (n-1)\left(\frac{S_a'(r)}{S_a(r)} - \frac{1}{r}\right)$$

Integrate in r $\Rightarrow$ $f(r) \le S_a(r)$. $\underset{\text{integrate again}}{\Rightarrow}$ DONE!

□